Published by Institute of Physics Publishing for SISSA/ISAS

Received: *May 14, 2003*
Revised: *May 16, 2003*
Accepted: *May 16, 2003*

Axial anomaly in the reduced model: higher representations

Teruaki Inagaki

Graduate School of Science and Engineering, Ibaraki University
Mito 310-8512, Japan
E-mail: teruaki@serra.sci.ibaraki.ac.jp

Yoshio Kikukawa

Department of Physics, Nagoya University
Nagoya 464-8602, Japan
E-mail: kikukawa@eken.phys.nagoya-u.ac.jp

Hiroshi Suzuki

Department of Mathematical Sciences, Ibaraki University
Mito 310-8512, Japan
E-mail: hsuzuki@mx.ibaraki.ac.jp

Abstract: The axial anomaly arising from the fermion sector of U(N) or SU(N) reduced model is studied under a certain restriction of gauge field configurations (the "U(1) embedding" with $N = L^d$). We use the overlap-Dirac operator and consider how the anomaly changes as a function of a gauge-group representation of the fermion. A simple argument shows that the anomaly vanishes for an irreducible representation expressed by a Young tableau whose number of boxes is a multiple of L^2 (such as the adjoint representation) and for a tensor-product of them. We also evaluate the anomaly for general gauge-group representations in the large N limit. The large N limit exhibits expected algebraic properties as the axial anomaly. Nevertheless, when the gauge group is SU(N), it does not have a structure such as the trace of a product of traceless gauge-group generators which is expected from the corresponding gauge field theory.

Keywords: Renormalization Regularization and Renormalons, Lattice Gauge Field Theories, Gauge Symmetry, Anomalies in Field and String Theories.

http://jhep.sissa.it/archive/papers/jhep052003042/jhep052003042.pdf

JHEP05(2003)042

Contents

1. Introduction

It has been unclear for long time how to define the topological charge in the reduced model for large N QCD [1]–[4]. One may try to define the topological charge in a "fermionic way" through the index theorem. However, since the reduced model is given by a zero-volume limit of a field theory, it is a system of finite degrees of freedom as long as N is finite even very large. It is then obvious that one cannot have the axial anomaly, unless a certain source of an explicit breaking of axial symmetry is introduced from an onset. Then one may ask: What is a good way to simulate the (quantum) axial symmetry breaking in a system of finite degrees of freedom?

Recently, motivated by a success of the overlap-Dirac operator [5] in lattice gauge theory (which is also a finite system when the lattice size is finite), authors of a paper [6] proposed a use of the overlap-Dirac operator in the quenched reduced model [2]. The overlap-Dirac operator satisfies the Ginsparg-Wilson relation [7] and this relation ensures remarkable properties concerning the chiral symmetry. For example, an index relation which is analogous to that in the continuum holds even with finite degrees of freedom [8]. Hence, the overlap Dirac operator provides a natural definition of the topological charge in the reduced model. In fact, explicit gauge configurations which have a non-trivial topological charge have been given in ref. [6].

A similar problem to define the topological charge (or the index) may be posed in the context of a matrix model for the type IIB superstring [9] which is a zero-dimensional reduction of the $N = 1$ super Yang-Mills theory in ten dimensions. In this context, a use of the overlap-Dirac operator or the overlap [10] has been proposed [11]. (More precisely, this is for a compact version of the IIB matrix model. See also ref. [12].) In a related context, a use of Ginsparg-Wilson relation has been actively investigated recently [13, 14].

In a paper [15], two of us studied chiral anomalies arising from a fundamental fermion in the "naive" [1] or the quenched [2] reduced model. There, it was pointed out that a certain restriction of reduced gauge fields (the U(1) embedding) allows a mapping of the problem to that of lattice gauge theory. By using available techniques in the latter, we determined a general form of the topological charge resulted by a use of the overlap-Dirac operator. Also, in chiral-gauge reduced models, it was shown that a single fundamental fermion gives rise to an obstruction for a smooth fermion integration measure, which is analogous to the gauge anomaly in the original theory before reduction.

An important question postponed in ref. [15] is how chiral anomalies (which was evaluated only for a fundamental fermion) change as a function of a gauge-group representation of the fermion. In particular, we are interested in if the anomaly cancellation in the original theory is realized in the chiral-gauge reduced model. As a step toward this investigation, in this paper we study the topological charge (or the axial anomaly) in the reduced model for various gauge-group representations.

In section 2, we present a general setting of our problem in the naive or the quenched reduced model. Next, in section 3, we recapitulate basic reasoning and results of ref. [15] concerning the axial anomaly. In section 4, on the basis of a structure of the overlap-Dirac operator, we present general properties of the axial anomaly which can be stated without any approximation. In subsequent sections, we perform a large N calculation of the axial anomaly. To illustrate the idea of this calculational scheme, we first re-derive the result of ref. [15] within this approximation. Then, in section 6, this scheme is applied to general gauge-group representations. Section 7 is devoted to the conclusion.

2. Reduced model with the overlap-Dirac operator

The fermion sector of the vector-like reduced model is defined by

$$\langle \mathcal{O} \rangle = \int \mathrm{d}\psi \mathrm{d}\overline{\psi}\, \mathcal{O} \exp(-\overline{\psi} D \psi)\,, \tag{2.1}$$

where $\mathcal{O}$ is an arbitrary operator containing fermion variables and D is a Dirac operator. In this paper, we use the overlap-Dirac operator [5] as D. The overlap-Dirac operator is given by

$$D = 1 - A(A^\dagger A)^{-1/2}\,, \qquad A = 1 - D_\mathrm{w}\,, \tag{2.2}$$

where D_w is the Wilson-Dirac operator:[1]

$$D_\mathrm{w} = \frac{1}{2}\left[\gamma_\mu(\nabla_\mu^* + \nabla_\mu) + \sum_\mu (\nabla_\mu^* - \nabla_\mu)\right]. \tag{2.3}$$

[1]We choose parameters in the Wilson-Dirac operator as $m_0 = r = 1$. The Greek indices, μ, ν, ..., runs over 1, 2, ..., d, where d denotes a dimensionality of the system which is assumed to be even.

JHEP05(2003)042

The covariant derivatives in this expression depend on the gauge-group representation R, to which the fermion belongs. For the fundamental representation which will be denoted by F, they read

$$\nabla_\mu \psi = U_\mu \psi - \psi \,, \qquad \nabla^*_\mu \psi = \psi - U^\dagger_\mu \psi \,, \tag{2.4}$$

where $U_\mu \in \mathrm{U}(N)$ or $\mathrm{SU}(N)$ is the reduced gauge field.

The above prescription for the gauge coupling correspond to the "naive" reduced model [1]. In the case of the quenched reduced model [2], the Dirac operator should be defined with a momentum insertion by the factor e^{ip_μ}. Since in this paper we will treat the gauge field as a non-dynamical background, this phase factor can be absorbed into the reduced gauge field without loss of generality. So we will omit this momentum factor in the following discussion.

A general representation of the gauge group $\mathrm{SU}(N)$ is represented by $(\psi)_{i_1,\ldots,i_n}$, where each of indices $i_1, \ldots, i_n$ transforms as the fundamental representation and indices may have certain symmetric properties, as represented by the Young tableau. For example, the adjoint fermion is expressed by $(\psi)_{i_1,i_2,\ldots,i_N}$ where last $N-1$ indices are totally anti-symmetric. This form of representation, however, has a certain limitation in applying our large N calculation, because our large N calculation is justified only when the number of indices is $O(N^0)$. In the above case of the adjoint fermion, we may equally express it as $(\psi)_{i_1;j_1}$ by contracting last $N-1$ indices with the invariant tensor so that the large N calculation is applied. With this situation in mind, we consider a general representation expressed by

$$(\psi)_{i_1,\ldots,i_n;j_1,\ldots,j_m} \,, \tag{2.5}$$

where each of indices $i_1, \ldots i_n$ transforms as the fundamental representation and each of $j_1, \ldots j_m$ transforms as the anti-fundamental representation. Indices may have certain symmetric properties. We denote the structure of eq. (2.5) as (n, m), irrespective of its symmetry with respect to indices. For a representation expressed by eq. (2.5), the covariant derivatives are defined by

$$\begin{aligned}(\nabla_\mu \psi)_{i_1,\ldots,i_n;j_1,\ldots,j_m} &= \\ &= (U_\mu)_{i_1 k_1} \cdots (U_\mu)_{i_n k_n} (\psi)_{k_1,\ldots,k_n;l_1,\ldots,l_m} (U^\dagger_\mu)_{l_1 j_1} \cdots (U^\dagger_\mu)_{l_m j_m} - (\psi)_{i_1,\ldots,i_n;j_1,\ldots,j_m} \,,\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}(\nabla^*_\mu \psi)_{i_1,\ldots,i_n;j_1,\ldots,j_m} &= \\ &= (\psi)_{i_1,\ldots,i_n;j_1,\ldots,j_m} - (U^\dagger_\mu)_{i_1 k_1} \cdots (U^\dagger_\mu)_{i_n k_n} (\psi)_{k_1,\ldots,k_n;l_1,\ldots,l_m} (U_\mu)_{l_1 j_1} \cdots (U_\mu)_{l_m j_m} \,.\end{aligned} \tag{2.7}$$

Now for the overlap-Dirac operator to be well-defined, we require that the gauge field is "admissible" [16, 6]. For the fundamental representation, the condition reads

$$\| 1 - U_\mu U_\nu U^\dagger_\mu U^\dagger_\nu \| < \epsilon \,, \tag{2.8}$$

where ϵ is a constant smaller than $(2-\sqrt{2})/d(d-1)$. For a fermion in a general representation, the plaquette variable in this condition is replaced by the plaquette in the corresponding gauge representation. In this paper, however, we always require the condition (2.8) for the fundamental representation, because we are interested in how the axial anomaly changes as a function of the gauge-group representation. Namely, we consider the anomaly for various gauge-group representations (including the fundamental representation) by taking the *same* gauge-field configuration as the gauge field background.

The overlap-Dirac operator satisfies the Ginsparg-Wilson relation [7]

$$\gamma_{d+1}D + D\gamma_{d+1} = D\gamma_{d+1}D\,, \tag{2.9}$$

which implies an exact symmetry of the fermion action [17]: The action $\overline{\psi}D\psi$ is invariant under substitutions, $\psi \to \psi + \delta\psi$ and $\overline{\psi} \to \overline{\psi} + \delta\overline{\psi}$, where

$$\delta\psi = i\hat{\gamma}_{d+1}\psi\,, \qquad \delta\overline{\psi} = i\overline{\psi}\gamma_{d+1}\,, \tag{2.10}$$

and $\hat{\gamma}_{d+1}$ is the modified chiral matrix defined by

$$\hat{\gamma}_{d+1} = \gamma_{d+1}(1-D), \qquad \hat{\gamma}_{d+1}^2 = 1\,. \tag{2.11}$$

The fermion integration measure on the other hand acquires a non-trivial jacobian Q_R (here the subscript R signifies the representation of the fermion):

$$\langle\delta\mathcal{O}\rangle = 2iQ_R\langle\mathcal{O}\rangle\,, \qquad Q_R = \frac{1}{2}\,\mathbf{Tr}(\gamma_{d+1} + \hat{\gamma}_{d+1})\,, \tag{2.12}$$

where **Tr** denotes a trace over a complete set which spans a space of ψ, including the trace over spinor indices. This combination Q_R is an integer [10, 8] again as a consequence of the Ginsparg-Wilson relation and the γ_5-hermiticity, $D^\dagger = \gamma_{d+1}D\gamma_{d+1}$. Hence we may regard this jacobian Q_R as a topological charge in the reduced model [6, 15].

3. U(1) embedding and admissible U(1) gauge fields[2]

In ref. [15], a general expression of the topological charge Q_F (here the subscript F signifies the fundamental representation) was given under a certain restriction of reduced gauge fields. This restriction is termed U(1) embedding and defined as follows. First we assume that N is d-th power of a certain integer L, $N = L^d$. Then the gauge field is assumed to have the form

$$U_\mu = u_\mu \Gamma_\mu\,, \tag{3.1}$$

where u_μ is an $N \times N$ *diagonal* matrix and Γ_μ is the "shift matrix"

$$\Gamma_\mu = \mathbf{1}_L \otimes \cdots \otimes \mathbf{1}_L \otimes U \otimes \mathbf{1}_L \otimes \cdots \otimes \mathbf{1}_L\,. \tag{3.2}$$

In this expression, $\mathbf{1}_L$ represents the $L \times L$ unit matrix and the factor U appears in the μ-th entry. The $L \times L$ unitary matrix U is given by

$$U = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}. \tag{3.3}$$

[2]This section gives a brief sketch of the basic reasoning of ref. [15]. For details, the readers are asked to refer to ref. [15].

When substituting eq. (3.1) in eq. (2.4), one finds that covariant derivatives take an identical form as covariant derivatives in the *conventional* lattice gauge theory. The size of the corresponding lattice Γ is L, $\Gamma = \{x \in \mathbb{Z}^d \mid 0 \leq x_\mu < L\}$, and a site $x = (x_1, \ldots, x_d)$ on the lattice and an index of the fundamental representation $1 \leq i \leq L^d = N$ are identified by

$$i(x) = 1 + x_d + Lx_{d-1} + \cdots + L^{d-1}x_1 \,. \tag{3.4}$$

Then the shift matrix Γ_μ realizes a shift on the lattice in the μ-th direction,

$$(\Gamma_\mu f \Gamma_\mu^\dagger)_{i(x)i(x)} = f_{i(x+\hat{\mu})i(x+\hat{\mu})} \,, \tag{3.5}$$

for an arbitrary *diagonal* matrix f. The gauge fields on the lattice is given by diagonal elements of the matrix u_μ. Since u_μ is unitary, all diagonal elements are pure-phase. Hence, the gauge field in the reduced model which has the particular form (3.1) is mapped to U(1) lattice gauge fields. Moreover, the plaquette variable in the reduced model is identical to the plaquette variable in the U(1) lattice theory:

$$U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger = u_\mu (\Gamma_\mu u_\nu \Gamma_\mu^\dagger)(\Gamma_\nu u_\mu^\dagger \Gamma_\nu^\dagger) u_\nu^\dagger \,, \tag{3.6}$$

which is a diagonal matrix and its diagonal element is the U(1) plaquette on Γ:

$$\begin{aligned}(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger)_{i(x)i(x)} &= (u_\mu)_{i(x)i(x)} (u_\nu)_{i(x+\hat{\mu})i(x+\hat{\mu})} (u_\mu)^*_{i(x+\hat{\nu})i(x+\hat{\nu})} (u_\nu)^*_{i(x)i(x)} \\ &= u_\mu(x) u_\nu(x+\hat{\mu}) u_\mu(x+\hat{\nu})^* u_\nu(x)^* \end{aligned} \tag{3.7}$$

due to eq. (3.5). In appendix A, we show that the above restriction (3.1) is nothing but the orbifolding [18].

With the U(1) embedding, we can thus utilize available techniques in the conventional U(1) lattice gauge theory. In particular, the above equivalence of plaquette variables shows that the condition for admissible gauge fields (2.8) is common in both matrix and lattice pictures. A complete parameterization of admissible U(1) lattice gauge fields has been known [19]. In terms of the lattice gauge theory on Γ, it is given by

$$u_\mu(x) = \omega(x) v_\mu^{[m]}(x) u_\mu^{[w]}(x) e^{i a_\mu^{\mathrm{T}}(x)} \omega(x+\hat{\mu})^* \,. \tag{3.8}$$

In this expression, each factor has the following meaning: The field $\omega(x) \in \mathrm{U}(1)$ is the U(1) gauge transformation (note that the admissibility (2.8) is a gauge invariant condition). The field $u_\mu^{[w]}(x)$ is defined by

$$u_\mu^{[w]}(x) = \begin{cases} w_\mu, & \text{for } x_\mu = 0, \\ 1, & \text{otherwise,} \end{cases} \qquad w_\mu \in \mathrm{U}(1) \,, \tag{3.9}$$

and has a vanishing field strength $f_{\mu\nu}(x) = 0$, where

$$f_{\mu\nu}(x) = \frac{1}{i} \ln u_\mu(x) u_\nu(x+\hat{\mu}) u_\mu(x+\hat{\nu})^* u_\nu(x)^* \,, \qquad -\pi < f_{\mu\nu}(x) \leq \pi \,. \tag{3.10}$$

However, the field $u_\mu^{[w]}(x)$ carries the Wilson (or Polyakov) line, $\prod_{s=0}^{L-1} u_\mu^{[w]}(s\hat{\mu}) = w_\mu \in \mathrm{U}(1)$. The field $v_\mu^{[m]}(x)$ is defined by

$$v_\mu^{[m]}(x) = \exp\left[-\frac{2\pi i}{L^2}\left(L\delta_{x_\mu,L-1} \sum_{\nu>\mu} m_{\mu\nu} x_\nu + \sum_{\nu<\mu} m_{\mu\nu} x_\nu\right)\right] \,, \tag{3.11}$$

and carries a constant field strength

$$f_{\mu\nu}(x) = \frac{2\pi}{L^2} m_{\mu\nu} \,, \tag{3.12}$$

where the "magnetic flux" $m_{\mu\nu}$ is an *integer* bounded by[3]

$$|m_{\mu\nu}| < \frac{\epsilon'}{2\pi} L^2 \,. \tag{3.13}$$

The "transverse" gauge potential $a_\mu^{\mathrm{T}}(x)$ is defined by[4]

$$\begin{aligned} &\partial_\mu^* a_\mu^{\mathrm{T}}(x) = 0 \,, \qquad \sum_{x\in\Gamma} a_\mu^{\mathrm{T}}(x) = 0 \,, \\ &|f_{\mu\nu}(x)| = \left| \partial_\mu a_\nu^{\mathrm{T}}(x) - \partial_\nu a_\mu^{\mathrm{T}}(x) + \frac{2\pi m_{\mu\nu}}{L^2} \right| < \epsilon' \,. \end{aligned} \tag{3.14}$$

Note that the space of $a_\mu^{\mathrm{T}}(x)$ is contractible. Namely, we may smoothly deform as $a_\mu^{\mathrm{T}}(x) \to 0$ without being against the admissibility.

The above is the U(1) embedding for the U(N) reduced model in which u_μ in eq. (3.1) has no restriction except that it must be a diagonal matrix. When the gauge group is SU(N), $\prod_{x\in\Gamma} u_\mu(x)$ must be unity. This requires that $w_\mu \in \mathbb{Z}_{L^{d-1}}$ and $\prod_{x\in\Gamma} v_\mu^{[m]}(x) = \exp[-\pi i L^{d-2}(L-1)\sum_\nu m_{\mu\nu}] = 1$. Note that the latter is always satisfied for $d > 2$.

When a local Dirac operator which obeys the Ginsparg-Wilson relation is employed, we can moreover invoke a powerful cohomological technique which determines a general structure of the axial anomaly in U(1) lattice gauge theory [20]. Under the U(1) embedding, we can apply this result in the lattice theory to the reduced model. In this way, we have for the fundamental representation [15]

$$Q_F = \frac{(-1)^{d/2}}{2^{d/2}(d/2)!} \epsilon_{\mu_1\nu_1\cdots\mu_{d/2}\nu_{d/2}} m_{\mu_1\nu_1} m_{\mu_2\nu_2} \cdots m_{\mu_{d/2}\nu_{d/2}} \,, \tag{3.15}$$

which is manifestly an integer. Within the U(1) embedding, this is the general form of Q_F defined by the overlap-Dirac operator.

Eq. (3.15) establishes that the space of reduced gauge fields, after the admissibility (2.8) is imposed, is divided into many topological sectors. In ref. [6], this fact has been shown by using the abelian background of ref. [21]. We emphasize that this topological structure is *not* due to our restriction of reduced gauge fields, the U(1) embedding (3.1). We obtained Q_F (3.15) for a subspace of the whole space of admissible reduced gauge fields, which can be expressed by eq. (3.1). This is enough to conclude a non-trivial topological structure of the space of admissible fields, because two configurations (which are expressed by eq. (3.1)) with different Q_F (3.15) cannot smoothly be connected even by relaxing the constraint (3.1), without affecting the admissibility (2.8). Recall that Q_F itself is defined for all admissible fields by eq. (2.12). The sole reason we stick to the U(1) embedding is that it allows a precise parameterization of admissible gauge fields (3.8).

[3] $\epsilon' = 2\arcsin(\epsilon/2)$.

[4] ∂_μ and ∂_μ^* denote the forward and the backward difference operators on Γ, $\partial_\mu f(x) = f(x+\hat\mu) - f(x)$, $\partial_\mu^* f(x) = f(x) - f(x-\hat\mu)$, respectively.

4. General properties of Q_R

For the fundamental representation, a trick of U(1) embedding works perfectly. If one is interested in the axial anomaly for other representations with the *same* gauge-field background, however, the U(1) embedding does not provide a useful picture such as the matrix-lattice correspondence. We have to find another kind of argument. In this section, we summarize general properties of Q_R which can be stated without any approximation. We also present a simple argument within the U(1) embedding that $Q_R = 0$ for a representation R with $n - m = 0 \bmod L^2$ in eq. (2.5).

We start with the expression

$$Q_R = \frac{1}{2}\,\mathbf{Tr}\,\frac{H}{\sqrt{H^2}}\,, \tag{4.1}$$

$$H = \gamma_{d+1}A = \gamma_{d+1}\left[-\frac{1}{2}\gamma_\mu(\nabla_\mu^* + \nabla_\mu) + 1 - \frac{1}{2}\sum_\mu(\nabla_\mu^* - \nabla_\mu)\right], \tag{4.2}$$

which follows from eqs. (2.12), (2.11) and (2.2). Since Q_R is given by a single trace over group indices, for a direct-sum representation $R_1 \oplus R_2$, we have

$$Q_{R_1\oplus R_2} = Q_{R_1} + Q_{R_2}\,. \tag{4.3}$$

Also for a complex conjugate representation R^*, we have[5]

$$Q_{R^*} = (-1)^{d/2} Q_R\,. \tag{4.4}$$

These are expected algebraic properties as the axial anomaly.

Now, we assume the U(1) embedding (3.1). By noting $\gamma_\mu\gamma_\nu = \delta_{\mu\nu} + [\gamma_\mu, \gamma_\nu]/2$, we have

$$\begin{aligned} H^2 =& -\frac{1}{4}(\nabla_\mu^* + \nabla_\mu)^2 + \left[1 - \frac{1}{2}\sum_\mu(\nabla_\mu^* - \nabla_\mu)\right]^2 - \\ & - \frac{1}{8}\gamma_\mu\gamma_\nu[\nabla_\mu^* + \nabla_\mu, \nabla_\nu^* + \nabla_\nu] - \frac{1}{4}\gamma_\mu\sum_\nu[\nabla_\mu^* + \nabla_\mu, \nabla_\nu^* - \nabla_\nu]. \end{aligned} \tag{4.5}$$

This shows that, in Q_R (4.1), the gamma matrices in the denominator $\sqrt{H^2}$ always accompany a commutator of covariant derivatives. A commutator of covariant derivatives can be computed from the definition (2.6). The result is

$$\begin{aligned} &([\nabla_\mu, \nabla_\nu]\psi)_{i_1,\ldots,i_n;j_1,\ldots,j_m} = \\ &= (U_\mu U_\nu)_{i_1k_1}\cdots(U_\mu U_\nu)_{i_nk_n}(\psi)_{k_1,\ldots,k_n;l_1,\ldots,l_m}(U_\nu^\dagger U_\mu^\dagger)_{l_1j_1}\cdots(U_\nu^\dagger U_\mu^\dagger)_{l_mj_m} - \\ &\quad - (U_\nu U_\mu)_{i_1k_1}\cdots(U_\nu U_\mu)_{i_nk_n}(\psi)_{k_1,\ldots,k_n;l_1,\ldots,l_m}(U_\mu^\dagger U_\nu^\dagger)_{l_1j_1}\cdots(U_\mu^\dagger U_\nu^\dagger)_{l_mj_m}\,. \end{aligned} \tag{4.6}$$

[5]This follows from the property of the charge conjugation matrix C in d-dimensional euclidean space. It commutes with γ_{d+1} in $d = 4k$ dimensions, while it anti-commutes with γ_{d+1} in $d = 4k + 2$ dimensions:

$$[C, \gamma_{d+1}] = 0 \quad (d = 4k)\,, \qquad \{C, \gamma_{d+1}\} = 0 \quad (d = 4k+2)\,.$$

See ref. [22] for example.

JHEP05(2003)042

This involves an exchange $U_\mu \leftrightarrow U_\nu$ which can be expressed by the plaquette variable as

$$U_\mu U_\nu = (U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger) U_\nu U_\mu = e^{if_{\mu\nu}} U_\nu U_\mu \,, \tag{4.7}$$

where we have used the relation (3.7) for the U(1) embedding and here $f_{\mu\nu}$ is a diagonal matrix whose diagonal elements are given by eq. (3.10) for the parameterization (3.8).

Since Q_R is an integer, it is invariant under a deformation of the gauge field, as long as the deformation is not against the admissibility (2.8). In particular, we may remove the degrees of freedom $a_\mu^{\mathrm{T}}(x)$ in eq. (3.8), because a deformation $a_\mu^{\mathrm{T}}(x) \to 0$ does not affect the admissibility. Then the field strength of admissible gauge fields is given by the *constant* (3.12). Namely, we can set

$$U_\mu U_\nu = e^{2\pi i m_{\mu\nu}/L^2} U_\nu U_\mu \,, \tag{4.8}$$

in evaluation of Q_R. This shows

$$\begin{aligned} &([\nabla_\mu, \nabla_\nu]\psi)_{i_1,\ldots,i_n;j_1,\ldots,j_m} = \\ &\quad = \left[e^{2\pi i(n-m)m_{\mu\nu}/L^2} - 1\right] \times \\ &\quad\quad \times (U_\nu U_\mu)_{i_1k_1} \cdots (U_\nu U_\mu)_{i_nk_n} (\psi)_{k_1,\ldots,k_n;l_1,\ldots,l_m} (U_\mu^\dagger U_\nu^\dagger)_{l_1j_1} \cdots (U_\mu^\dagger U_\nu^\dagger)_{l_mj_m} \,, \end{aligned} \tag{4.9}$$

and similar considerations show

$$\begin{aligned} &([\nabla_\mu, \nabla_\nu^*]\psi)_{i_1,\ldots,i_n;j_1,\ldots,j_m} = \\ &\quad = \left[1 - e^{-2\pi i(n-m)m_{\mu\nu}/L^2}\right] \times \\ &\quad\quad \times (U_\nu^\dagger U_\mu)_{i_1k_1} \cdots (U_\nu^\dagger U_\mu)_{i_nk_n} (\psi)_{k_1,\ldots,k_n;l_1,\ldots,l_m} (U_\mu^\dagger U_\nu)_{l_1j_1} \cdots (U_\mu^\dagger U_\nu)_{l_mj_m} \,, \end{aligned} \tag{4.10}$$

and

$$\begin{aligned} &([\nabla_\mu^*, \nabla_\nu^*]\psi)_{i_1,\ldots,i_n;j_1,\ldots,j_m} = \\ &\quad = \left[e^{2\pi i(n-m)m_{\mu\nu}/L^2} - 1\right] \times \\ &\quad\quad \times (U_\nu^\dagger U_\mu^\dagger)_{i_1k_1} \cdots (U_\nu^\dagger U_\mu^\dagger)_{i_nk_n} (\psi)_{k_1,\ldots,k_n;l_1,\ldots,l_m} (U_\mu U_\nu)_{l_1j_1} \cdots (U_\mu U_\nu)_{l_mj_m} \,. \end{aligned} \tag{4.11}$$

As a consequence, we see that commutators of covariant derivatives vanish when $n - m = 0 \bmod L^2$.

As a side remark, we note that the factor $e^{2\pi i(n-m)m_{\mu\nu}/L^2}$ appearing in above expressions is the plaquette variable in the representation R.[6] This factor is simply proportional to the identity matrix and when the gauge group is SU(N), this implies that the plaquette has its value in the center of SU(N). When L is large, the plaquette behaves as $1 + 2\pi i(n-m)m_{\mu\nu}/L^2$ which does *not* have a structure as one plus (traceless) Lie algebra valued matrix, although the plaquette itself belongs to SU(N). An implication of this fact will be commented later.

[6]Hence these expressions show that the configuration (3.8) with $a_\mu^{\mathrm{T}}(x) = 0$ in fact satisfies the admissibility condition for a representation R, if $|[(n-m) \bmod L^2] m_{\mu\nu}| < \epsilon' L^2/(2\pi)$.

JHEP05(2003)042

As we have noted, if commutators of covariant derivatives vanish, we have no gamma matrices in the denominator of eq. (4.1), $\sqrt{H^2}$. Then we cannot have an enough number of gamma matrices to have non-zero trace with respect to spinor indices in Q_R. Therefore we conclude

$$Q_R = 0\,, \qquad \text{for} \qquad n - m = 0 \pmod{L^2}\,. \tag{4.12}$$

For an irreducible representation of $\mathrm{SU}(N)$ expressed by the Young tableau, the condition $n - m = 0 \bmod L^2$ implies that the total number of boxes is a multiple of L^2. In particular, since the adjoint representation corresponds to $(n,m) = (1,1)$ (or equivalently $(n,m) = (N,0)$), $Q_R = 0$ for the adjoint fermion within the $\mathrm{U}(1)$ embedding. A tensor-product representation of such irreducible representations also has $n - m = 0 \bmod L^2$. So it also leads $Q_R = 0$.

5. Q_F in the large N limit

It seems difficult to conclude something exact about Q_R for general gauge-group representations than eq. (4.12). In what follows, we will consider an approximate treatment in the large N or equivalently the large L limit. Our scheme was inspired by a calculation of ref. [14]. Although our target is general gauge-group representations, it is helpful to consider first the fundamental representation in this limit to illustrate the idea.

First to give a precise meaning of the trace in eq. (4.1), we have to introduce some complete basis for the fermion variable. For this, we use eigenvectors of the shift matrix

$$\Gamma_\mu \phi(\vec{p}) = e^{2\pi i p_\mu / L} \phi(\vec{p})\,, \qquad \{\, \vec{p} \in \mathbb{Z}^d \mid 0 \le p_\mu < L \,\}\,. \tag{5.1}$$

The explicit form of $\phi(\vec{p})$ is given by

$$\phi(\vec{p}) = \frac{1}{\sqrt{N}} \hat{\phi}(\vec{p}) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{5.2}$$

where the $N \times N$ matrix $\hat{\phi}(\vec{p})$ is defined by

$$\hat{\phi}(\vec{p}) = V^{p_1} \otimes \cdots \otimes V^{p_d}\,, \qquad V = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{L-1} \end{pmatrix}, \tag{5.3}$$

with $\omega = e^{2\pi i / L}$. One can easily verify eq. (5.1) and

$$\phi(\vec{p})^\dagger \phi(\vec{q}) = \delta_{\vec{p},\vec{q}}\,. \tag{5.4}$$

It can also be shown that $\phi(\vec{p})$ span a complete basis (appendix B). So we have

$$\sum_{\vec{p}} \phi(\vec{p}) \phi(\vec{p})^\dagger = \mathbf{1}_N\,. \tag{5.5}$$

JHEP05(2003)042

A fermion in the fundamental representation can then be expanded in this basis as

$$\psi_i = \sum_{\vec{p}} \phi(\vec{p})_i c(\vec{p}) \,. \tag{5.6}$$

The topological charge (4.1) for the fundamental fermion is then given by

$$Q_F = \frac{1}{2} \sum_{\vec{p}} \phi(\vec{p})^\dagger \frac{H}{\sqrt{H^2}} \phi(\vec{p}) \,. \tag{5.7}$$

Now, our large N calculation of Q_F proceeds as follows. First, for configurations in the U(1) embedding (3.1) with eq. (3.8), we set $\omega(x) = 1$ and $a_\mu^{\mathrm{T}}(x) = 0$. These choices do not affect Q_F because Q_F is gauge invariant and the integer Q_F does not change under the deformation $a_\mu^{\mathrm{T}}(x) \to 0$ that is consistent with the admissibility. Next, we assume that in eq. (3.8),

$$u_\mu = \mathbf{1}_N \left(1 + O\left(\frac{1}{L}\right)\right) , \tag{5.8}$$

or equivalently

$$U_\mu = \Gamma_\mu \left(1 + O\left(\frac{1}{L}\right)\right) . \tag{5.9}$$

From the explicit form of admissible gauge fields, we see that the assumption is fulfilled except for the factor $u_\mu^{[w]}(x)$ (the Wilson-line degrees of freedom) of eq. (3.9). We note however that for a fixed value of the Wilson line w_μ, one may choose a different representation of $u_\mu^{[w]}(x)$ by making use of the gauge degrees of freedom. A possible choice is

$$u_\mu^{[w]}(x) = (w_\mu)^{1/L} , \tag{5.10}$$

which in fact reproduces an identical Wilson line $\prod_{s=0}^{L-1} u_\mu^{[w]}(s\hat{\mu}) = w_\mu$. Then the condition (5.8) is fulfilled for eq. (5.10). Once the condition (5.9) matches, it is straightforward to find the large N limit (or the large L limit) of Q_F.

Recall the structure of H^2 (4.5). Noting that eqs. (4.9)–(4.11) show that commutators of covariant derivatives are proportional to $2\pi i m_{\mu\nu}/L^2$ in the large L limit, we see that the leading term in the large N limit is contained in an expansion of

$$\begin{aligned} Q_F = &\frac{1}{2} \sum_{\vec{p}} \operatorname{tr} \gamma_{d+1} \left[-i\gamma_\mu s_\mu + 1 + \sum_\mu (c_\mu - 1) \right] \times \\ &\times \left\{ s_\nu^2 + \left[1 + \sum_\nu (c_\nu - 1) \right]^2 - \frac{2\pi i m_{\nu\rho}}{2L^2} \gamma_\nu \gamma_\rho c_\nu c_\rho + \frac{2\pi i m_{\nu\rho}}{L^2} \gamma_\nu i c_\nu s_\rho \right\}^{-1/2} , \end{aligned} \tag{5.11}$$

where we have noted eqs. (5.9) and (5.1) and used abbreviations

$$s_\mu = \sin \frac{2\pi}{L} p_\mu \,, \qquad c_\mu = \cos \frac{2\pi}{L} p_\mu \,. \tag{5.12}$$

The expression (5.11) is familiar in the *classical continuum limit* of the axial anomaly which is defined by the overlap-Dirac operator in the conventional lattice gauge theory [23]. See eq. (12) of ref. [24]. The lattice spacing a in the latter is formally identified with $1/L$ in our expression. The correspondence becomes perfect in the large N limit, in which

$$k_\mu = \frac{2\pi}{L} p_\mu \,, \tag{5.13}$$

can be regarded as a continuous momentum and thus the sum over $\vec{p}$ becomes momentum integration

$$\sum_{\vec{p}} \to L^d \int_{-\pi}^{\pi} \frac{\mathrm{d}^d k}{(2\pi)^d} \,. \tag{5.14}$$

Therefore we can literally copy the result of ref. [24]. The large N limit of Q_F is then given by

$$Q_F = \frac{(-1)^{d/2}}{2^{d/2}(d/2)!} \epsilon_{\mu_1\nu_1\cdots\mu_{d/2}\nu_{d/2}} m_{\mu_1\nu_1} m_{\mu_2\nu_2} \cdots m_{\mu_{d/2}\nu_{d/2}} \,, \tag{5.15}$$

which coincides with eq. (3.15). Eq. (3.15) was originally obtained for finite N by using the local cohomology technique and here we computed it in the large N limit. However, since Q_F (3.15) is independent of N, a coincide with the large N limit (5.15) is expected. Nevertheless, this coincidence is assuring: In fact the above large N calculation might be regarded as another proof of eq. (3.15).

6. Q_R in the large N limit

An advantage of the large N approach is that it is applicable to higher representations for which an application of the local cohomology technique is difficult.[7] First, it is straightforward to generalize the above large N calculation for the fundamental representation to a tensor-product of (anti-)fundamental representations. A fermion in the tensor-product representation can be expanded in an analogous way as eq. (5.6)

$$\begin{aligned}(\psi)_{i_1,\ldots,i_n;j_1,\ldots,j_m} = \sum_{\vec{p}_1} \cdots \sum_{\vec{p}_n} \sum_{\vec{q}_1} \cdots \sum_{\vec{q}_m} \phi(\vec{p}_1)_{i_1} \cdots \phi(\vec{p}_n)_{i_n} \phi(\vec{q}_1)^\dagger_{j_1} \cdots \phi(\vec{q}_m)^\dagger_{j_m} \times \\ \times\, c(\vec{p}_1, \ldots, \vec{p}_n; \vec{q}_1, \ldots, \vec{q}_m) \,.\end{aligned} \tag{6.1}$$

The topological charge (4.1) for a (n,m)-type tensor-product is then given by

$$\begin{aligned}Q_{(n,m)} = \frac{1}{2} \sum_{\vec{p}_1} \cdots \sum_{\vec{p}_n} \sum_{\vec{q}_1} \cdots \sum_{\vec{q}_m} \phi(\vec{p}_1)^\dagger_{i_1} \cdots \phi(\vec{p}_n)^\dagger_{i_n} \phi(\vec{q}_1)_{j_1} \cdots \phi(\vec{q}_m)_{j_m} \times \\ \times \left(\frac{H}{\sqrt{H^2}}\right)_{i_1,\ldots,i_n;j_1,\ldots,j_m;k_1,\ldots,k_n;l_1,\ldots,l_m} \times \\ \times\, \phi(\vec{p}_1)_{k_1} \cdots \phi(\vec{p}_n)_{k_n} \phi(\vec{q}_1)^\dagger_{l_1} \cdots \phi(\vec{q}_m)^\dagger_{l_m} \,.\end{aligned} \tag{6.2}$$

[7] By using techniques in refs. [25], it is possible to map the fermion sector of the reduced model to a noncommutative field theory. However, with a general lack of locality in noncommutative field theories, it seems rather difficult to develop a local cohomological argument in these theories.

The following steps are almost identical as for the fundamental representation. Noting eqs. (4.9)–(4.11), we find the expression (5.11) with the following substitution:

$$\sum_{\vec{p}} \to \sum_{\vec{p}_1} \cdots \sum_{\vec{p}_n} \sum_{\vec{q}_1} \cdots \sum_{\vec{q}_m}, \tag{6.3}$$

and

$$m_{\mu\nu} \to (n-m) m_{\mu\nu}\,, \tag{6.4}$$

and

$$\begin{aligned} s_\mu &\to \sin \frac{2\pi}{L} (\vec{p}_1 + \cdots + \vec{p}_n - \vec{q}_1 - \cdots - \vec{q}_m)_\mu\,, \\ c_\mu &\to \cos \frac{2\pi}{L} (\vec{p}_1 + \cdots + \vec{p}_n - \vec{q}_1 - \cdots - \vec{q}_m)_\mu\,. \end{aligned} \tag{6.5}$$

Here we have assumed that n and m are of $O(N^0)$, i.e., they are not large numbers. This assumption eliminates a possibility that a large number of link variables in a covariant derivative compensate a "weakness" of gauge fields in the large N limit (5.9). Then the argument for the fundamental representation with above substitutions proceeds as it stands.

In eq. (5.11) with above substitutions, we shift the variable, say $\vec{p}_1$, as $\vec{p}_1 \to \vec{p}_1 - \vec{p}_2 \cdots - \vec{p}_n + \vec{q}_1 + \cdots + \vec{q}_m$ so that the "integrand" becomes independent of $\vec{p}_2$, …, $\vec{p}_n$, $\vec{q}_1$, …, $\vec{q}_m$. The summation over these variables can then be done and it gives a factor

$$\sum_{\vec{p}_2} \cdots \sum_{\vec{p}_n} \sum_{\vec{q}_1} \cdots \sum_{\vec{q}_m} 1 = (L^d)^{n+m-1} = N^{n+m-1}\,. \tag{6.6}$$

The summation over $\vec{p}_1$ becomes the integral (5.14) in the large N limit. In this way, we have

$$Q_{(n,m)} = (n-m)^{d/2} N^{n+m-1} Q_F (1 + O(1/L))\,, \qquad \text{for} \qquad n, m = O(N^0)\,. \tag{6.7}$$

Note that $N^{n+m-1} = (\dim R)/N$ for the tensor-product representation R.

With the above experience, let us consider the large N calculation for higher irreducible representations. For general irreducible representation R, the topological charge (4.1) can be expressed as

$$\begin{aligned} Q_{(n,m)} = \frac{1}{2} & \sum_{\vec{p}_1} \cdots \sum_{\vec{p}_n} \sum_{\vec{q}_1} \cdots \sum_{\vec{q}_m} \phi(\vec{p}_1)^\dagger_{i_1} \cdots \phi(\vec{p}_n)^\dagger_{i_n} \phi(\vec{q}_1)_{j_1} \cdots \phi(\vec{q}_m)_{j_m} \times \\ & \times \left(\frac{H}{\sqrt{H^2}} \mathcal{P}_R \right)_{i_1,\ldots,i_n;j_1,\ldots,j_m;k_1,\ldots,k_n;l_1,\ldots,l_m} \times \\ & \times \phi(\vec{p}_1)_{k_1} \cdots \phi(\vec{p}_n)_{k_n} \phi(\vec{q}_1)^\dagger_{l_1} \cdots \phi(\vec{q}_m)^\dagger_{l_m}\,, \end{aligned} \tag{6.8}$$

where $\mathcal{P}_R$ is the projection operator for the irreducible representation R. For the anti-symmetric representation, for example, it reads

$$(\mathcal{P}_R)_{i_1 i_2;;k_1 k_2;} = \frac{1}{2} (\delta_{i_1 k_1} \delta_{i_2 k_2} - \delta_{i_1 k_2} \delta_{i_2 k_1})\,. \tag{6.9}$$

For other irreducible representations (that can be obtained by appropriate (anti-) symmetrization of indices of a tensor-product representation), it is straightforward to construct $\mathcal{P}_R$. In our large N calculation, we do not need the explicit form of $\mathcal{P}_R$; what we need is the weight w of the "identity operator" in $\mathcal{P}_R$:

$$\begin{aligned}(\mathcal{P}_R)_{i_1,\ldots,i_n;j_1,\ldots,j_m;k_1,\ldots,k_n;l_1,\ldots,l_m} &= \\ &= w\delta_{i_1k_1}\cdots\delta_{i_nk_n}\delta_{j_1l_1}\cdots\delta_{j_ml_m} + \left(\widetilde{\mathcal{P}}_R\right)_{i_1,\ldots,i_n;j_1,\ldots,j_m;k_1,\ldots,k_n;l_1,\ldots,l_m}, \end{aligned} \tag{6.10}$$

where the operator $\widetilde{\mathcal{P}}_R$ exchanges at least one index-pair. To find a value of w, we note the relation

$$\begin{aligned}\sum_{\vec{p}_1}\cdots\sum_{\vec{p}_n}\sum_{\vec{q}_1}\cdots\sum_{\vec{q}_m}&\phi(\vec{p}_1)^\dagger_{i_1}\cdots\phi(\vec{p}_n)^\dagger_{i_n}\phi(\vec{q}_1)_{j_1}\cdots\phi(\vec{q}_m)_{j_m}\times \\ &\times(\mathcal{P}_R)_{i_1,\ldots,i_n;j_1,\ldots,j_m;k_1,\ldots,k_n;l_1,\ldots,l_m}\times \\ &\times\phi(\vec{p}_1)_{k_1}\cdots\phi(\vec{p}_n)_{k_n}\phi(\vec{q}_1)^\dagger_{l_1}\cdots\phi(\vec{q}_m)^\dagger_{l_m} = \\ &= \dim R \\ &= wN^d\left(1+O\left(\frac{1}{N}\right)\right),\end{aligned} \tag{6.11}$$

where $\dim R$ is the dimension of the representation R. This relation holds because the operator $\widetilde{\mathcal{P}}_R$ produces a delta function of a pair of momenta which restricts the "phase-space" integral by a factor N. Therefore, we have

$$w = \frac{\dim R}{N^d}\left(1+O\left(\frac{1}{N}\right)\right). \tag{6.12}$$

Now, when acting on $\phi(\vec{p}_1)_{k_1}\cdots\phi(\vec{p}_n)_{k_n}\phi(\vec{q}_1)^\dagger_{l_1}\cdots\phi(\vec{q}_m)^\dagger_{l_m}$, the projection operator $\mathcal{P}_R$ exchanges the name of "wave functions" within the set $\{\phi(\vec{p}_1),\ldots,\phi(\vec{p}_n)\}$ and within $\{\phi(\vec{q}_1)^\dagger,\ldots,\phi(\vec{q}_m)^\dagger\}$, but not extending these two sets. Since the action of $H/\sqrt{H^2}$ (in the large N limit) is invariant under this exchange, as can be seen from eq. (6.5), what is left after the same argument as before is

$$\begin{aligned}Q_F = \frac{1}{2}\sum_{\vec{p}_1}\cdots\sum_{\vec{p}_n}\sum_{\vec{q}_1}\cdots\sum_{\vec{q}_m}&\phi(\vec{p}_1)^\dagger_{i_1}\cdots\phi(\vec{p}_n)^\dagger_{i_n}\phi(\vec{q}_1)_{j_1}\cdots\phi(\vec{q}_m)_{j_m}\times \\ &\times(\mathcal{P}_R)_{i_1,\ldots,i_n;j_1,\ldots,j_m;k_1,\ldots,k_n;l_1,\ldots,l_m}\,\phi(\vec{p}_1)_{k_1}\cdots\phi(\vec{p}_n)_{k_n}\phi(\vec{q}_1)^\dagger_{l_1}\cdots\phi(\vec{q}_m)^\dagger_{l_m}\times \\ &\times\operatorname{tr}\gamma_{d+1}\left[-i\gamma_\mu s_\mu + 1 + \sum_\mu(c_\mu - 1)\right]\times \\ &\times\left\{s_\nu^2 + \left[1+\sum_\nu(c_\nu-1)\right]^2 - \frac{2\pi i(n-m)m_{\nu\rho}}{2L^2}\gamma_\nu\gamma_\rho c_\nu c_\rho + \right. \\ &\left. + \frac{2\pi i(n-m)m_{\nu\rho}}{L^2}\gamma_\nu i c_\nu s_\rho\right\}^{-1/2},\end{aligned} \tag{6.13}$$

where s_μ and c_μ are given by eq. (6.5). By substituting eq. (6.10), the operator $\widetilde{\mathcal{P}}_R$ produces only a sub-leading contribution of $O(1/N)$. So, by only taking a contribution of the identity operator, the integration over (say) $\vec{p}_1$ can be done in the large N limit as before. In this way, we finally have[8]

$$Q_R = (n-m)^{d/2}\frac{\dim R}{N}Q_F\left(1+O\left(\frac{1}{L}\right)\right), \qquad \text{for} \qquad n,m=O(N^0)\,. \tag{6.14}$$

This large N result in fact reproduces our previous results; eq. (3.15) when $(n,m)=(1,0)$, eq. (4.12) when $n=m$ and eq. (6.7) for tensor-product representations. One can express this anomaly in an analogous form to the Chern character: Introduce a 2-form

$$M_R = \frac{1}{2}(n-m)_R\, m_{\mu\nu}\, \mathrm{d}x_\mu \mathrm{d}x_\nu\,, \tag{6.15}$$

then

$$Q_R\, \mathrm{d}^d x = \frac{1}{N}\operatorname{tr}_R \exp(-M_R)\Big|_{\mathrm{d}^d x}\left(1+O\left(\frac{1}{L}\right)\right). \tag{6.16}$$

We note algebraic properties of M_R, $M_{R_1\otimes R_2}=M_{R_1}+M_{R_2}$ and $M_{R^*}=-M_R$ ($R\to R^*$ is equivalent to an exchange $n\leftrightarrow m$), which is analogous to that of the field strength 2-form. Our Q_R in the large N limit thus satisfies eq. (4.4). For a direct-product of two representations, we have

$$\begin{aligned} Q_{R_1\otimes R_2}\mathrm{d}^d x &= \frac{(-1)^{d/2}}{(d/2)!}\frac{1}{N}\operatorname{tr}_{R_1\otimes R_2}(M_{R_1}+M_{R_2})^{d/2} \\ &= Q_{R_1}\dim R_2\, \mathrm{d}^d x + Q_{R_2}\dim R_1\, \mathrm{d}^d x + \cdots\,. \end{aligned} \tag{6.17}$$

This is an expected algebraic property as the axial anomaly. Recall however that with the U(1) embedding, the plaquette of the reduced gauge field has its value in the center of SU(N). So even when the gauge group is SU(N), $\operatorname{tr}_R M_R\neq 0$. The axial anomaly we found thus has a different structure from the axial anomaly in the continuum SU(N) gauge field theory, in which the anomaly is given by the trace of a product of traceless gauge-group generators.

7. Conclusion

In this paper, we have studied the axial anomaly arising from the fermion sector of the naive or the quenched reduced model. We used the overlap-Dirac operator and consider a restriction of reduced gauge fields by U(1) embedding. Our main results are eq. (4.12) and eq. (6.14) for the axial anomaly for general gauge-group representations. We expect that our analyses in this paper will be useful to investigate a possible gauge anomaly in the reduced model [15] for general gauge-group representations and its cancellation among fermion multiplet.

[8] An interesting observation which we have verified for a wide class of gauge-group representations is that the coefficient $(n-m)\dim R/N$ is an *integer*. We have no proof of this statement at the moment. If this generally holds, we would speculate $Q_R=(n-m \bmod L^2)^{d/2}(\dim R)Q_F/N$ is an exact expression being valid even for finite N.

Acknowledgments

H.S. would like to thank Tohru Eguchi, Kazuo Fujikawa, Takanori Fujiwara, Yusuke Taniguchi and Ke Wu for useful discussions. This work was completed when we attended RIKEN topical meeting, "Chiral fermion on the lattice and geometry of matrix models". We thank the organizer and participants, in particular, Masashi Hayakawa, Satoshi Iso, Hikaru Kawai and Tsukasa Tada, for their warm hospitality and discussions. This work is supported in part by Grant-in-Aid for Scientific Research, #12640262, #14046207 (Y.K.) and #13740142 (H.S.).

A. U(1) embedding is the orbifolding[9]

We define a diagonal matrix f

$$f = e^{-\pi i L(N-1)/N^2} \Lambda^{L^{d-1}} \otimes \Lambda^{L^{d-2}} \otimes \cdots \otimes \Lambda^{L} \otimes \Lambda \,, \tag{A.1}$$

where

$$\Lambda = \begin{pmatrix} 1 & & & & \\ & \lambda & & & \\ & & \lambda^2 & & \\ & & & \ddots & \\ & & & & \lambda^{L-1} \end{pmatrix}, \qquad \lambda = e^{2\pi i/N} = e^{2\pi i/L^d}. \tag{A.2}$$

In eq. (A.1), the factor $e^{-\pi i L(N-1)/N^2}$ is multiplied so that $\det f = 1$. Namely, $f \in \mathrm{SU}(N)$. In the notation of eq. (3.4), the diagonal element of f is given by

$$f_{i(x)i(x)} = e^{-\pi i L(N-1)/N^2} \lambda^{i(x)-1} \,. \tag{A.3}$$

Since the integer $i(x)-1$ runs over all integers from 0 to $N-1$, all diagonal elements of f are distinct pure-phase. Also, one can verify

$$\Gamma_\mu f \Gamma_\mu^\dagger = \lambda^{L^{d-\mu}} f \,. \tag{A.4}$$

Now we impose a constraint on the reduced gauge field[10]

$$U_\mu = \lambda^{L^{d-\mu}} f U_\mu f^\dagger \,. \tag{A.5}$$

In terms of a matrix u_μ defined by $U_\mu = u_\mu \Gamma_\mu$, this reads

$$u_\mu = f u_\mu f^\dagger \,. \tag{A.6}$$

Since f is a diagonal matrix whose all elements are distinct pure-phase, this implies that u_μ is a diagonal matrix. Namely, the constraint (A.5) is equivalent to the U(1) embedding (3.1). In eq. (A.5), the conjugation by $f \in \mathrm{SU}(N)$ can be regarded as the gauge

[9] This fact was suggested to us by Yusuke Taniguchi. A similar observation from a more general view point has been presented in ref. [26]. An observation that a matrix model with appropriate constraints can be regarded as a lattice theory dates back to the early study on the reduced model [27].

[10] Note however that in the U(1) embedding we do not place any restriction on reduced fermion fields.

JHEP05(2003)042

transformation in the reduced model and a multiplication of the factor $\lambda^{L^{d-\mu}} \in \mathrm{U}(1)$ can be regarded as $\mathrm{U}(1)^d$ transformation, under which the pure-gauge reduced model (i.e., the plaquette action) is invariant [1]–[4]. In this sense, the constraint (A.5) can be regarded as the orbifold projection [18]. Namely, the U(1) embedding is a simple example of the orbifolding. Among the $\mathrm{U}(N)$ or $\mathrm{SU}(N)$ gauge transformations in the reduced model, $U_\mu \to \Omega U_\mu \Omega^\dagger$, those which are consistent with the projection (A.5) survive as the gauge symmetry in the U(1) embedding. These are given by diagonal Ω's and are nothing but U(1) gauge transformations in the U(1) embedding, discussed in ref. [15].

B. $\phi(\vec{p})$ spans a complete basis

Let y be an integer, $y = 1, 2, \ldots, L-1$. Then $(\omega^y)^L = 1$ but $\omega^y \neq 1$. Defining a polynomial $P(M) = \sum_{p=0}^{L-1} M^p/L$, these imply

$$P(\omega^y) = 0, \qquad P(1) = 1\,. \tag{B.1}$$

Then an inspection of the structure of V shows

$$P(V\omega^{-x}) = \mathrm{diag}(0, \ldots, 0, 1, 0, \ldots, 0),, \qquad x = 0, 1, \ldots, L-1, \tag{B.2}$$

where the non-zero element appears in the $x+1$-th entry. Then we see that the N-vector

$$P(V\omega^{-x_1}) \otimes \cdots \otimes P(V\omega^{-x_d}) \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \frac{1}{N} \sum_{\vec{p}} \phi(\vec{p}) e^{-2\pi i \vec{p}\cdot\vec{x}/L}\,, \tag{B.3}$$

where $x_\mu = 0, 1, \ldots, L-1$, has a unique non-zero element ($= 1$) in the $i(x)$-th entry, where $i(x)$ is given by eq. (3.4). Since this collection of vectors forms a complete basis, $\phi(\vec{p})$ does too.

References

[1] T. Eguchi and H. Kawai, *Reduction of dynamical degrees of freedom in the large-N gauge theory*, *Phys. Rev. Lett.* **48** (1982) 1063.

[2] G. Bhanot, U.M. Heller and H. Neuberger, *The quenched Eguchi-Kawai model*, *Phys. Lett.* **B 113** (1982) 47;
D.J. Gross and Y. Kitazawa, *A quenched momentum prescription for large-N theories*, *Nucl. Phys.* **B 206** (1982) 440;
H. Levine and H. Neuberger, *A quenched reduction for the topological limit of QCD*, *Phys. Lett.* **B 119** (1982) 183.

[3] A. Gonzalez-Arroyo and M. Okawa, *A twisted model for large-N lattice gauge theory*, *Phys. Lett.* **B 120** (1983) 174; *The twisted Eguchi-Kawai model: a reduced model for large-N lattice gauge theory*, *Phys. Rev.* **D 27** (1983) 2397;
T. Eguchi and R. Nakayama, *Simplification of quenching procedure for large-N spin models*, *Phys. Lett.* **B 122** (1983) 59;
S.R. Das, *Quark fields in twisted reduced large-N QCD*, *Phys. Lett.* **B 132** (1983) 155.

[4] S.R. Das, *Some aspects of large N theories*, *Rev. Mod. Phys.* **59** (1987) 235.

[5] H. Neuberger, *Exactly massless quarks on the lattice*, *Phys. Lett.* **B 417** (1998) 141 [hep-lat/9707022]; *More about exactly massless quarks on the lattice*, *Phys. Lett.* **B 427** (1998) 353 [hep-lat/9801031].

[6] J. Kiskis, R. Narayanan and H. Neuberger, *Proposal for the numerical solution of planar QCD*, *Phys. Rev.* **D 66** (2002) 025019 [hep-lat/0203005].

[7] P.H. Ginsparg and K.G. Wilson, *A remnant of chiral symmetry on the lattice*, *Phys. Rev.* **D 25** (1982) 2649;
P. Hasenfratz, *Prospects for perfect actions*, *Nucl. Phys.* **63** *(Proc. Suppl.)* (1998) 53 [hep-lat/9709110]; *Lattice QCD without tuning, mixing and current renormalization*, *Nucl. Phys.* **B 525** (1998) 401 [hep-lat/9802007].

[8] P. Hasenfratz, V. Laliena and F. Niedermayer, *The index theorem in QCD with a finite cut-off*, *Phys. Lett.* **B 427** (1998) 125 [hep-lat/9801021].

[9] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, *A large-N reduced model as superstring*, *Nucl. Phys.* **B 498** (1997) 467 [hep-th/9612115];
H. Aoki et al., *IIB matrix model*, *Prog. Theor. Phys. Suppl.* **134** (1999) 47 [hep-th/9908038].

[10] R. Narayanan and H. Neuberger, *Infinitely many regulator fields for chiral fermions*, *Phys. Lett.* **B 302** (1993) 62 [hep-lat/9212019]; *Chiral determinant as an overlap of two vacua*, *Nucl. Phys.* **B 412** (1994) 574 [hep-lat/9307006]; *Chiral fermions on the lattice*, *Phys. Rev. Lett.* **71** (1993) 3251 [hep-lat/9308011]; *A construction of lattice chiral gauge theories*, *Nucl. Phys.* **B 443** (1995) 305 [hep-th/9411108];
P.Y. Huet, R. Narayanan and H. Neuberger, *Overlap formulation of Majorana–Weyl fermions*, *Phys. Lett.* **B 380** (1996) 291 [hep-th/9602176];
S. Randjbar-Daemi and J. Strathdee, *On the overlap formulation of chiral gauge theory*, *Phys. Lett.* **B 348** (1995) 543 [hep-th/9412165]; *Chiral fermions on the lattice*, *Nucl. Phys.* **B 443** (1995) 386 [hep-lat/9501027]; *On the overlap prescription for lattice regularization of chiral fermions*, *Nucl. Phys.* **B 466** (1996) 335 [hep-th/9512112]; *Consistent and covariant anomalies in the overlap formulation of chiral gauge theories*, *Phys. Lett.* **B 402** (1997) 134 [hep-th/9703092].

[11] N. Kitsunezaki and J. Nishimura, *Unitary IIB matrix model and the dynamical generation of the space time*, *Nucl. Phys.* **B 526** (1998) 351 [hep-th/9707162].

[12] T. Tada and A. Tsuchiya, *Toward a supersymmetric unitary matrix formulation of the IIB matrix model*, *Prog. Theor. Phys.* **103** (2000) 1069 [hep-th/9903037].

[13] H. Aoki, S. Iso and K. Nagao, *Ginsparg-Wilson relation, topological invariants and finite noncommutative geometry*, *Phys. Rev.* **D 67** (2003) 085005 [hep-th/0209223];
A.P. Balachandran and G. Immirzi, *The fuzzy Ginsparg-Wilson algebra: a solution of the fermion doubling problem*, hep-th/0301242.

[14] S. Iso and K. Nagao, *Chiral anomaly and Ginsparg-Wilson relation on the noncommutative torus*, hep-th/0212284.

[15] Y. Kikukawa and H. Suzuki, *Chiral anomalies in the reduced model*, *J. High Energy Phys.* **09** (2002) 032 [hep-lat/0207009].

JHEP05(2003)042

[16] P. Hernandez, K. Jansen and M. Lüscher, *Locality properties of Neuberger's lattice Dirac operator*, *Nucl. Phys.* **B 552** (1999) 363 [hep-lat/9808010];
H. Neuberger, *Bounds on the Wilson Dirac operator*, *Phys. Rev.* **D 61** (2000) 085015 [hep-lat/9911004].

[17] M. Lüscher, *Exact chiral symmetry on the lattice and the Ginsparg-Wilson relation*, *Phys. Lett.* **B 428** (1998) 342 [hep-lat/9802011].

[18] M.R. Douglas and G.W. Moore, *D-branes, quivers and ALE instantons*, hep-th/9603167;
S. Kachru and E. Silverstein, *4d conformal theories and strings on orbifolds*, *Phys. Rev. Lett.* **80** (1998) 4855 [hep-th/9802183];
M. Schmaltz, *Duality of non-supersymmetric large-N gauge theories*, *Phys. Rev.* **D 59** (1999) 105018 [hep-th/9805218];
M.J. Strassler, *On methods for extracting exact non-perturbative results in non-supersymmetric gauge theories*, hep-th/0104032;
D.B. Kaplan, E. Katz and M. Unsal, *Supersymmetry on a spatial lattice*, hep-lat/0206019;
A.G. Cohen, D.B. Kaplan, E. Katz and M. Unsal, *Supersymmetry on a euclidean spacetime lattice, I. A target theory with four supercharges*, hep-lat/0302017.

[19] M. Lüscher, *Abelian chiral gauge theories on the lattice with exact gauge invariance*, *Nucl. Phys.* **B 549** (1999) 295 [hep-lat/9811032].

[20] M. Lüscher, *Topology and the axial anomaly in abelian lattice gauge theories*, *Nucl. Phys.* **B 538** (1999) 515 [hep-lat/9808021];
T. Fujiwara, H. Suzuki and K. Wu, *Non-commutative differential calculus and the axial anomaly in abelian lattice gauge theories*, *Nucl. Phys.* **B 569** (2000) 643 [hep-lat/9906015]; *Axial anomaly in lattice abelian gauge theory in arbitrary dimensions*, *Phys. Lett.* **B 463** (1999) 63 [hep-lat/9906016];
H. Suzuki, *Anomaly cancellation condition in lattice gauge theory*, *Nucl. Phys.* **B 585** (2000) 471 [hep-lat/0002009];
H. Igarashi, K. Okuyama and H. Suzuki, *Errata and addenda to "Anomaly cancellation condition in lattice gauge theory"*, hep-lat/0012018;
Y. Kikukawa and Y. Nakayama, *Gauge anomaly cancellations in* $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ *electroweak theory on the lattice*, *Nucl. Phys.* **B 597** (2001) 519 [hep-lat/0005015];
M. Lüscher, *Lattice regularization of chiral gauge theories to all orders of perturbation theory*, *J. High Energy Phys.* **06** (2000) 028 [hep-lat/0006014];
Y. Kikukawa, *Domain wall fermion and chiral gauge theories on the lattice with exact gauge invariance*, *Phys. Rev.* **D 65** (2002) 074504 [hep-lat/0105032];
H. Igarashi, K. Okuyama and H. Suzuki, *More about the axial anomaly on the lattice*, *Nucl. Phys.* **B 644** (2002) 383 [hep-lat/0206003].

[21] L. Giusti, A. Gonzalez-Arroyo, C. Hoelbling, H. Neuberger and C. Rebbi, *Fermions on tori in uniform abelian fields*, *Phys. Rev.* **D 65** (2002) 074506 [hep-lat/0112017].

[22] L. Alvarez-Gaumé, *An introduction to anomalies*, HUTP-85/A092, *Lectures given at Int. School on Mathematical Physics*, Erice, Italy, Jul 1–14 1985.

[23] Y. Kikukawa and A. Yamada, *Weak coupling expansion of massless QCD with a Ginsparg-Wilson fermion and axial* U(1) *anomaly*, *Phys. Lett.* **B 448** (1999) 265 [hep-lat/9806013];
K. Fujikawa, *A continuum limit of the chiral jacobian in lattice gauge theory*, *Nucl. Phys.* **B 546** (1999) 480 [hep-th/9811235];

JHEP05(2003)042

D.H. Adams, *Axial anomaly and topological charge in lattice gauge theory with overlap-Dirac operator*, *Ann. Phys. (NY)* **296** (2002) 131 [hep-lat/9812003]; *On the continuum limit of fermionic topological charge in lattice gauge theory*, *J. Math. Phys.* **42** (2001) 5522 [hep-lat/0009026];
H. Suzuki, *Simple evaluation of chiral jacobian with the overlap Dirac operator*, *Prog. Theor. Phys.* **102** (1999) 141 [hep-th/9812019];
T.-W. Chiu and T.-H. Hsieh, *Perturbation calculation of the axial anomaly of Ginsparg-Wilson fermion*, hep-lat/9901011;
T. Reisz and H.J. Rothe, *The axial anomaly in lattice QED: a universal point of view*, *Phys. Lett.* **B 455** (1999) 246 [hep-lat/9903003];
M. Frewer and H.J. Rothe, *Universality of the axial anomaly in lattice QCD*, *Phys. Rev.* **D 63** (2001) 054506 [hep-lat/0004005].

[24] T. Fujiwara, K. Nagao and H. Suzuki, *Axial anomaly with the overlap-Dirac operator in arbitrary dimensions*, *J. High Energy Phys.* **09** (2002) 025 [hep-lat/0208057].

[25] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162];
J. Ambjørn, Y.M. Makeenko, J. Nishimura and R.J. Szabo, *Finite N matrix models of noncommutative gauge theory*, *J. High Energy Phys.* **11** (1999) 029 [hep-th/9911041]; *Nonperturbative dynamics of noncommutative gauge theory*, *Phys. Lett.* **B 480** (2000) 399 [hep-th/0002158]; *Lattice gauge fields and discrete noncommutative Yang-Mills theory*, *J. High Energy Phys.* **05** (2000) 023 [hep-th/0004147];
J. Nishimura and M.A. Vazquez-Mozo, *Noncommutative chiral gauge theories on the lattice with manifest star-gauge invariance*, *J. High Energy Phys.* **08** (2001) 033 [hep-th/0107110].

[26] H. Neuberger, *Topological effects in matrix models representing lattice gauge theories at large-N*, hep-th/0212097.

[27] M. Okawa, *Monte Carlo study of the Eguchi-Kawai model*, *Phys. Rev. Lett.* **49** (1982) 353.